0A Pictorial History of some Gravitational Instantons *

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Abstract

Four-dimensional Euclidean spaces that solve Einstein's equations are interpreted as WKB approximations to wavefunctionals of quantum geometry. These spaces are represented graphically by suppressing inessential dimensions and drawing the resulting figures in perspective representation of three-dimensional space, some of them stereoscopically. The figures are also related to the physical interpretation of the corresponding quantum processes.

1. Introduction

Understanding General Relativity means to a large extent coming to terms with its most important ingredient, geometry. Among his many contributions, Charlie has given us new variations of this theme [1], fascinating because geometry is so familiar on two-dimensional surfaces, but so remote from intuition on higher-dimensional spacetimes. The richness he uncovered is shown nowhere better than in the 137 figures of his masterful text [2].

Today quantum gravity [3] leads to new geometrical features. One of these is a new role for Riemannian (rather than Lorentzian) solutions of the Einstein field equations: such "instantons" can describe in WKB approximation the tunneling transitions that are classically forbidden, for example because they correspond to a change in the space's topology. In order to gain a pictorial understanding of these spaces we can try to represent the geometry as a whole with less important dimensions suppressed; an alternative is to follow the ADM method and show a history of the tunneling by slices of codimension one.

We can readily go from equation to picture thanks to computer plotting routines, from the simpler ones as incorporated in spreadsheet programs [4] to the more powerful

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versions of *Mathematica*.[1] We therefore decided it would be fun to see how well the computer can draw pictures associated with tunneling and instantons. In Section 2 we recall the idea behind these by an example of a two-dimensional potential. In the following sections we present and interpret several general relativity instantons, both sliced and unsliced.

2. Tunneling in Several Dimensions

The one-dimensional potential (Fig. 1a)

$$V(x) = x^2 - x^3 \tag{1}$$

is typical of the class to which tunneling arguments are often applied. The exponential

Figure 1. (a) Plot of the potential $V(x)$ with a resonance at $E_{res} = 0.06605$. (Here we have put $\hbar^2/m = .01$.) (b) Virtual state wavefunction $\psi(x)$ in this potential.

"barrier penetration" coefficient, which governs the probability of escape for a particle that is initially trapped near $x = 0$, is related to the "resonance" (virtual state) solution of the time-independent Schrödinger equation in this potential (Fig. 1b). The maximum of this wavefunction occurs in the trapping region near the relative minimum of the potential, and the oscillating "tail" in the exterior region is very small compared to this maximum. This property characterizes the virtual state. If the potential barrier is sufficiently high, virtual states occur at rather well-defined energies.

A priori the virtual state has nothing directly to do with the tunneling problem: it is stationary, real-valued (at $t = 0$) and yields no net flux into or out of the potential's trapping region. (These same properties also make this wavefunction easy to draw!) To obtain a wavefunction with nonzero flux we need to use neighboring energy states.

[1] *Mathematica* is a trademark of Wolfram Research Inc., 100 Trade Center Drive, Champaign, Il 61820-7237, USA. We also used the program *showstereo.m* available by e-mail from mathsource@wri.com. We thank Peter Hübner (MPI-Astro) and Dr.-Ing. Werner Rupp (DASA) for help with computer resources

The properties of their wavefunctions change rapidly as the energy is varied away from resonance: to first order in the variation the exterior phase changes, and to second order the interior amplitude decreases. Fig. 2 shows two nearby energy states whose exterior phase differs by $\pi/2$. From these we can build an outgoing wave. Because

Figure 2. Wavefunctions ψ_1 (dashed) and ψ_2 (dash-dotted) at energies $E_1 = 0.0663$ slightly above resonance, and at $E_2 = 0.0658$ slightly below resonance, showing the rapid phase change. The combination $\psi_1 - i\psi_2$ has a net outgoing flux that is also shown (solid).

the amplitude of these waves is essentially the same as that of the virtual state, the tail of the virtual state is a measure of the outgoing flux. Of course this measure is rather crude, giving us only the main exponential factor in the decay probability. Just how long it takes for the wave to leak out depends on how rapidly the properties of the wavefunction change with energy; thus this "prefactor" to the exponential is not determined by the virtual state alone, and is rather more difficult to compute.

In one dimension there is only one way for the particle to leak out of the potential in Fig. 1a (namely, toward positive x). To find in more detail "how" the particle gets out of the trapping region, and the analogous issues in tunneling of fields, we must consider at least two-dimensional potentials. Usually one treats the rotationally symmetric case: angular momentum is then conserved, and the problem reduces to a one-dimensional effective potential motion for each angular momentum. Because of the symmetry no direction of tunneling away from the trapping region is preferred over any other. But when the potential is not symmetric, some tunneling "paths" can be considerably more probable than others.[2]

A simple two-dimensional potential, in which the classical and the quantum mechanical problem can be solved because the corresponding equations are separable, is given

[2] An optical analogon is the observation of "frustrated total internal reflection" [5]. The potential barrier can be provided by the air space between the hypotenuse faces of two 90° prisms. More light (and at larger wavelengths) gets through where this barrier is narrowest.

by

$$V(x,y) = x^2 + y^2 - x^3 - 0.8y^3 = V_1(x) + V_2(y). \tag{2}$$

(Here the factor 0.8 was chosen only for convenience of plotting.) Fig. 3 shows a plot of this potential. It has a trapping region that is separated from the "exterior"

Figure 3. 3D plot of the potential $V(x,y)$. This, and some of the subsequent figures, are stereoscopic pairs similar to those found in the text *Methods of Theoretical Physics* by Morse and Feshbach. We know no better instructions how to view these figures than those found in the preface of that text.

region ($x \gg 0, y \gg 0$) by a barrier with two saddle points of different heights. The virtual state in this potential is simply the product,

$$\psi(x,y) = \psi_1(x)\psi_2(y)$$

where ψ_i is the virtual state in potential V_i. This two-dimensional wave function is plotted in Fig. 4a. In Fig. 4b the square of the wave function is indicated by increasing gray levels. We note that there is a path, namely along the x-axis, on which the wavefunction is generally (and particularly at the end of the exponential decrease) larger than along other paths leading from the trapping region to the exterior. This is the "most probable escape path" [6] that may be said to describe "how" the particle most likely tunnels through the barrier, in this sense: if the system were prepared in the virtual state and a position measurement were made in the barrier region, the particle would most likely be found near the most probable escape path. (Of course, after the measurement the particle would no longer be in the virtual state.)

The tunneling behavior of a virtual state is well approximated by a WKB wavefunction, and the analogous problem in *classical* mechanics [7]. For example, the most probable escape path is given by the "bounce" solution of particle motion in *imaginary time* — or equivalently, real time motion in the upside-down potential. The reader is encouraged to consider Fig. 3a turned upside down (for the less agile reader

Figure 4. (a) 3D log square plot of the virtual state wavefunction for the potential of Fig. 3. As in Fig. 2, the interference maxima and minima have little to do with the physical outgoing wave, in the case that this wavefunction is used to represent the decay of a virtual state. (b) Gray-level plot of the square of the wavefunction, showing the "most probable escape path" as a locus of relatively high density crossing the barrier region along the x-axis.

we have provided Fig. 5) and to imagine how a particle released at the central maximum would move in the resulting potential. Fig. 6 shows some of the orbits in such a potential. The bounce solutions are those that have a strict turning point, where the particle velocity vanishes; thus they are the only ones that reach the exterior zero equipotential, where energy conservation allows the particle that started at rest in the center to stop tunneling and again "become a classical particle."

Figure 5. Fig. 3 turned upside down.

Figure 6. The equipotentials of $-V(x, y)$ and some particle orbits in this potential.

Along the bounce path we can solve a one-dimensional time-independent Schrödinger equation to find the fall-off of the wavefunction, and hence obtain the tail amplitude and the exponential factor in the decay rate. To essentially the same approximation we can use the WKB estimate of this factor via the classical "Euclidean" action S_E of the bounce path (in the potential $-V$). Since this is to be computed for a motion in imaginary ("Euclidean") time, the action $S = iS_E$ itself is imaginary, and the lowest order WKB wavefunction, $e^{iS} = e^{-S_E}$ becomes a decreasing exponential. If we compute the action for the complete bounce, from the origin to the escape point and back to the origin, we obtain twice the action for the most probable escape path, which when exponentiated gives the probability (rather than the amplitude).

In fact, we can get a continuous picture of the particle's history if we allow the time parameter to change between real and imaginary at turning points. Fig. 7 shows such a picture of the nonrelativistic penetration history of the barrier of Eq. (1) (but with time plotted upward, as usual in relativity). We note that the infinite imaginary time needed for the tunneling has nothing to do with the real time needed for the physical process (which is vanishingly short in this approximation). The WKB wavefunction is given by the action evaluated along this entire history. The Euclidean part of the motion contributes an imaginary part to the action, and the real-time motion contributes a real part. Thus the WKB wavefunction exhibits exponential decrease in the Euclidean region, and oscillation in the real time region. (However,

Figure 7. The semiclassical history of the decay from the virtual state of the 1D potential $V(x)$. The particle starts at the classical ground state at $x = 0$, but moves in *imaginary* time, it, starting at $it = -\infty$, $x = \cosh^{-2}(it/\sqrt{2m})$. (This motion does not correspond to any passage of real, physical time, but signals the probabilistic nature of the process.) At the bounce (or "nucleation") time $it = 0$ the motion stops momentarily at $x = 1$, and then continues in real time, $x = \cos^{-2}(t/\sqrt{2m})$. In real time the particle seems to appear suddenly at $x = 1$, and then continues on a classical escape orbit.

this wavefunction does not have continuous derivatives at the turning points; this problem is usually solved by finding "transition formulas", but we neglect it in this lowest approximation.)

In general, as in our example, there may be more than one bounce and probable escape path. Because the probability is an exponential, the one with the least Euclidean action is generally overwhelmingly more probable (however, also see [8]). Of course, in order to get a non-zero decay probability, this least action must be finite (this is part of the definition of a bounce or instanton solution). Of the three bounces shown in Fig. 6, the diagonal one has the largest action (most improbable). In fact, it is really a combination of the other two bounces, rather than a different way for the particle to escape. This can be seen from the behavior of nearby particle orbits: they converge toward this bounce, showing that there is a second zero mode in the second variation of the action (the first zero mode is given by an infinitesimal time translation). Because the orbits intersect, the second variations vanish at the turning points, so that there will be two variations that lower the action. Only bounces with *one* negative mode should be counted as independent decay channels.

Another kind of Euclidean solution ("instanton") is useful for potentials in which the escape region is replaced by a second trapping region whose minimum is degenerate with the first (Fig. 8a). If one starts out with a wavepacket concentrated in one of the regions, it will at a later time be concentrated in the other region and vice versa, so that in general it has a fluctuating behavior. The two lowest energy eigenstates that most importantly contribute to this behavior are approximated by a sum resp. a difference between two WKB wavefunctions (of the type e^{S_E} and e^{-S_E}). These wavefunctions can again be evaluated by finding the action of a Euclidean solution of the classical equations of motion. But this instanton does not exhibit a bounce. Instead the particle takes an infinite imaginary time to move away from the center of the first region, *and* an infinite time to reach the center of the second region (Fig. 8b). A corresponding two-dimensional potential and its lowest energy wavefunction is shown in Fig. 9a. Again the wavefuntion is relatively large on the instanton path, making it the "most probable connecting path" (Fig. 9b).

Thus, in a multidimensional setting we may again say that the instanton tells us "how" the particle gets from one region to the other during a fluctuation. The instanton action will again give us the main exponential factor in the probability that a particle initially present in one region will be found in the other. In particular, the existence of an instanton with finite action indicates that the fluctuation in question does take place. For the details of the fluctuation, such as its frequency, one would again need information about more than one energy eigenstate.[3]

[3]In this connection, fluctuation means a transition between two or several states that are "classically allowed" and have the same energy. We do not mean the kind of virtual fluctuation that produces, for example, a virtual pair in the vacuum state. (One could however say that the barrier makes the virtual fluctuation real by preventing an immediate return to the initial state.)

Figure 8. (a) The potential $V(x) = -x^2 + x^4$ has two degenerate minima, separated by a tunneling region. (b) The semiclassical history of the tunneling in this potential, $x = (1/\sqrt{2})\tanh(it/\sqrt{m})$. The particle starts from one of the minima at $it = -\infty$, and reaches the other minimum at $it = \infty$, with the entrire motion taking place in imaginary time.

One assumes that an analogous semiclassical approximation is valid in field theory. That is, to see whether a classically forbidden transition is possible in the quantum theory one tries to find a finite action solution of the Euclidean field equations that connects the relevant classical initial and final states of the transition. If the connection is by a bounce solution, in which the initial state is reached only asymptotically, but the final state occurs at the turn-around "point" (surface of imaginary-time symmetry, or nucleation surface [9]), the transition is interpreted as a decay. If the connection is by an instanton (without a bounce), in which both the initial and final states are reached only asymptotically, the transition is a fluctuation. In general relativity, Euclidean solutions of the Einstein equations are Riemannian (rather than Lorentzian) manifolds.

3. Gravitational Bounces

If the classical initial state for tunneling has symmetry, the WKB tunneling state may or may not exhibit the same symmetry. If the Euclidean equations of motion admit a tunneling solution with the same symmetry, we expect this solution to have the lowest action. (If the symmetry is broken by the tunneling, then there is no unique instanton of smallest action, and one should sum over all of them.) For example, a

Figure 9. (a) A two-dimensional potential, $V(x, y) = -x^2 + x^4 + y^2$ with two degenerate minima. (b) The ground state wavefunction is relatively large along the most probable connecting path.

constant electric field is invariant under boosts in the field direction, and the instanton describing pair production by this field [10] has the same invariance. The electric field is also invariant under translation, but the instanton is not. However, the tunneling events related by translation are all equally probable.

In accordance with this expectation the "bubbles" of true vacuum expanding into false vacuum, both of which vacua are Lorentz invariant, do exhibit invariance under the homogeneous Lorentz group, and the corresponding Euclidean solutions is invariant under the rotation group $O(4)$. A simple example of a similar gravitational situation is provided by the "tunneling from nothing" into a deSitter universe [11], shown in Fig. 10 (with two dimensions suppressed). As in Fig. 7, in the lower part

Figure 10. The semiclassical history of deSitter space, in the same spirit as Fig. 7.

time is imaginary, and in the upper part it is real. Each part solves the equation $G_{\mu\nu} = \Lambda g_{\mu\nu}$ on its appropriate, Riemannian resp. Lorentzian, manifold. The nucleation surface is the equator; reflection of either part about this surface would give a complete manifold of each type, either a Riemannian bounce (a complete 4-sphere) or a complete Lorentzian deSitter universe (that bounces at some *minimum* radius).

To obtain a history of the deSitter evolution we normally slice the upper part of Fig. 10 by horizontal, spacelike surfaces. For a history of the tunneling it is natural to slice the lower part similarly by horizontal planes. This history indeed shows nothing as long as the plane is below the "south pole" of the hemisphere; when the plane touches the pole, the universe originates as a point, then expands into increasing 3-spheres (represented in the figure by circles) until it reaches the maximum radius of the virtual evolution, which is also the minimum radius of the deSitter space, and the real evolution starts.

We have tacitly assumed that Fig. 10 is to be rotated about two other axes to get the 4-manifolds (4-sphere, deSitter space) that we really intended. But we can just as well imagine the product of Fig. 10 with S^2, to generate the Nariai metric [12] by tunneling from nothing. In this case the nucleation surface, represented by the same horizontal plane in Fig. 10 as before, has the topology $S^2 \times S^1$ of a wormhole universe. The Euclidean part of the action turns out to be larger (corresponding to smaller tunneling probability) than for tunneling into the more symmetric deSitter space [9] (cf. the $y-$axis vs. the $x-$axis bounce of Fig. 6).

Further examples of $O(4)$ symmetric bounces are associated with vacuum decay. Since the ordinary 4-dimensional Minkowski space vacuum is stable [13], we have to consider compactified higher-dimensional cases. To be realistic the spacetime should be at least 5-dimensional, but in our figures we will have to suppress at least two of these dimensions: each point will represent a 2-sphere (with metric $d\Omega^2$ if it is a unit 2-sphere).[4]

Figure 11a shows the "5-dimensional Schwarzschild instanton,"

$$ds^2 = (1 - (R/r)^2)d\phi^2 + (1 - (R/r)^2)^{-1}dr^2 + r^2 d\Theta^2 + r^2 \sin^2\Theta d\Omega^2, \quad r \geq R \quad (3)$$

used by Witten [14] to discuss the decay of the Kaluza-Klein vacuum, with ϕ the compactification direction. (Since the picture is only qualitatively correct, it could also represent the 4-dimensional Schwarzschild instanton, which is usually taken to describe the thermal properties of a Schwarzschild black hole, but which in this connection would describe the decay of the 4-dimensional vacuum, compactified in one direction [15].) In the lower part of this figure we have plotted r and Θ in a vertical plane, and ϕ in the orthogonal horizontal direction. Since the latter is to be identified with period 2π, the front and back boundaries of the Figure are to be identified. That this is possible without singularity is shown in Fig. 11b, where this identification is performed on a horizontal slice of Fig. 11a, resulting in a "test tube" that is smoothly closed on one end, which represents the metrically correct embedding of the r, ϕ section of this metric.[5] The upper part of the Figure shows the analytic continuation to a Lorentzian manifold.

The surface $\Theta = \pi/2$ in the Witten instanton (3) (or any rotation of it) is a nucleation surface; it represents the final state of the tunneling. Since the instanton is asymptotically flat, the initial state, which it reaches asymptotically, is the vacuum. Once we have chosen a nucleation surface, the initial vacuum is most appropriately represented by the hyperplane at large distance from the origin that does not intersect the nucleation surface. One can then fill in other sections between these that give a reasonable tunneling history [16]. At first the topology of these sections is $\mathbb{R}^3 \times S^1$, like that of a spacelike surface of the compactified vacuum. When the minimum r on such a section reaches R, the section represents the singular instance at which the

[4]Actually, points related by symmetry about the vertical plane perpendicular to the paper correspond to the same 2-sphere.

[5]Since the test tube is topologically $\mathbb{R}^2$, the whole metric of Eq. (3) is $\mathbb{R}^2 \times S^3$, but one should remember that the $\mathbb{R}^2$ does not have the standard infinity, but instead becomes cylindrical — like the test tube.

Figure 11. (a) The semiclassical history of the decay of a compactified "false" 5D vacuum geometry is represented by the 3D region bounded above by the surface shown. Each point of this region represents a two-sphere, obtained by rotating in two additional dimensions about the center of the figure. The front and back parts of the surface are to identified in such a way that points along the middle curve C are regular (non-conical) origins at $r = R$ in the r, ϕ plane. (b) The actual geometry obtained from a slice of Fig. 11a by a plane perpendicular to the paper, with the identification carried out.

topology changes. Later sections have the topology $\mathbb{R}^2 \times S^2$, like that of the final state. In the Figure this looks like two disconnected spaces moving apart, but they are of course connected through the two dimensions that were suppressed[4], so that in fact a spherical hole has appeared in the final state, and expands to infinity in the subsequent Lorentzian development.

As a final example we show the creation of two magnetically charged Wheeler wormhole mouths by the magnetic field in an initially Melvin universe [17]. This remarkable topology-changing bounce has two axes of symmetry: one is the line joining the wormholes, and the other corresponds to invariance under boosts of the created pair. We suppress the former to reduce the Euclidean space dimensions to three, and show only the latter as a symmetry of the three-dimensional space in which the figure is drawn, namely the rotational symmetry of Fig. 12a about the horizontal axis, part of which is labeled "horizon." (Except for this symmetry the Figure is qualitative only.[6]) For ease of visualization we show only the lower, Euclidean half of the history — the space within the rectangular box and outside the semispherical cavities. As in the other figures, the upper part should consist of the analytic continuation of the lower half to a Lorentzian spacetime. Points on the inner surface of the two cavities are to be identified by reflection about the mid-plane. The top horizontal plane is the nucleation surface. It contains the axis of symmetry, where the rotational Killing vector vanishes. The part between the cavities is the wormhole's (Euclidean) Killing horizon. Its Lorentzian extension will as usual consist of two lightlike horizon surfaces, on which the boost Killing vector that is the continuation of the rotational Killing vector is lightlike. (The part of the axis outside the cavities, not drawn in the Figure, corresponds to the "Rindler horizon" that separates the two created wormhole mouths.)

On each horizontal section of Fig. 12a imagine a vector field like that representing a laminar fluid flow from left to right, avoiding the cavities. This can be taken as a representation of the magnetic field.[7] In the bottom surface this field is approximately constant and represents the field lines of the (asymptotic) Melvin universe's central region [19]; away from this central region the Melvin magnetic flux actually falls off with distance. At the nucleation surface some of this flux crosses the wormhole horizon and so has become trapped in the Wheeler wormhole. To show this in more detail we have drawn in Fig. 12b only the top, nucleation surface, but with the two circular curves in it identified as required. The wormhole's Killing horizon now occurs, as expected, at the narrowest part (throat) of the wormhole.

It is now not difficult to imagine other horizontal slices of Fig. 12a to obtain the rest of the Euclidean history of the wormhole formation. The singular slice is the one that first touches the cavities, so that only a pair of points are to be identified, rather than two circular curves. It is the first slice in which some flux has been trapped, and this trapped flux is conserved in the subsequent development. This flux is maximal in the sense that, after the wormhole mouths have moved far enough apart in their Lorentzian development to be compared to single black holes, they correspond to extremal Reissner-Nordström black holes. It is interesting that the spacelike distance through the wormhole is finite on the nucleation surface; it increases

[6] For another way of representing this geometry — in which two dimensions are suppressed — see Banks et al. [18].

[7] In a spacetime the electromagnetic field tensor should in general be represented by a honeycomb structure [2], not by lines. But once one has chosen a "spacelike" surface, the usual electric and magnetic field lines of course make sense.

Figure 12. (a) The imaginary-time part of the history of Wheeler wormhole production. Beyond the top (nucleation) surface the real-time history could be represented in 2+1 Minkowski space, with the rotational symmetry replaced by a boost symmetry. (b) The wormhole at the nucleation surface is the geometry of the top surface of part (a), with the identification of the two circles carried out. (These circles are the vertical, circular grid line that is partically hidden behind the wormhole's throat.) The grid lines can be taken to be equipotentials and field lines of the wormhole's field; to them should be added those of the Melvin universe background to get the net equipotentials and fields.

as the wormhole mouths move apart after nucleation, and asymptotically approaches the infinite value that one expects from the "horn" structure of the isolated Reissner-Nordström black hole [20]. It seems that other interesting features of the Lorentzian analytic continuation have been explored only to a limited extent, and they are too far removed from the instanton to be treated here [18].

4. Fluctuation Instantons

The solution of the Euclidean Einstein-Maxwell equations,

$$ds^2 = V^{-2}dt^2 + V^2(dx^2 + dy^2 + dz^2) \qquad V = \sum m_i/r_i, \qquad *F = d(V^{-1}) \wedge dt \quad (4)$$

can be interpreted as an instanton [21]. Here r_i, $i = 1...n$ are the distances in flat Euclidean 3-space from n different origins to the field point. This solution is similar to the multi-extremal Reissner-Nordström solution, except that it has no asymptotically flat region; instead it becomes cylindrical in the limit $r \to \infty$, just as it does in each "horn", $r_i \to 0$. If we place all the origins in the xy plane and suppress the $z-$ and $t-$directions, the geometry of the remaining dimensions can be shown embedded in flat Euclidean space as in Fig. 13. (The Gaussian curvature of this 2-surface

Figure 13. Imaginary time history of a fluctuation by which a maximally charged black hole throat splits into two.

is negative, reaching zero asymptotically; it appears remarkably difficult to find an accurate and unique embedding of such a surface, so Fig. 13 is correct only in its main features.) The way *Mathematica* sliced this figure suggests that this is an instanton that interpolates between a single universe and two or several daughter (or baby) universes. Indeed, the geometry and fields of Eq. (4) approach the Euclidean version of a Bertotti-Robinson universe [22] in all asymptotic regions. The fact that there is no bounce or nucleation surface, and that all connections to Lorentzian solutions are made in the asymptotic regions, indicates that this is a fluctuation-type instanton.

Many of the details of this instanton have been given elsewhere [21], where it is argued that it should also describe the quantum fluctuations of the region near the horizon of an extremal Reissner-Nordström black hole. Here we want to show the result of slicing this instanton for $n = 2$ and $m_1 = m_2$ by 3-surfaces. A simple choice consists of the surfaces on which V is constant. These slices are shown in Fig. 14. They are metrically accurate (except for the suppression of the axis direction). For small V, and hence large r, these surfaces of constant V have the topology of a single sphere. For large V, one or the other of the r_i has to be small, so one obtains two separate spheres. The critical, singular surface occurs when $V = m/d$, where d is the distance in Euclidean 3-space between the two origins. Depending on the interpretation, Fig. 14 then gives the imaginary-time history of the break-up of a Bertotti-Robinson universe, or of an extremal Reissner-Nordström black hole's horizon.

5. Conclusions

When instanton solutions were first investigated in general relativity, they were regarded primarily as a mathematical device, not to be interpreted in the same way as Lorentzian solutions — it was a bold step even to draw a Riemannian and a Lorentzian manifold connected in the same diagram. Since then it has become clear that many of the traditional methods work in both arenas. In particular, we have seen above that slicing by 3-dimensional surfaces can help the physical interpretation, just as it did for classical relativity when ADM pioneered this method.

Figure 14. (next page) Slices of constant V of the instanton of Fig. 13, with only one direction suppressed, (a) "before" the topology change (b) singular slice at the instant of topology change (c) "after" the topology change.

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